

# ON MODULUS OF NONCOMPACT CONVEXITY FOR A STRICTLY MINIMALIZABLE MEASURE OF NONCOMPACTNESS

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**ABSTRACT.** In this paper we consider modulus of noncompact convexity  $\Delta_{X,\phi}$  associated with the strictly minimalizable measure of noncompactness  $\phi$ . We also give some its properties and show its continuity on the interval  $[0, \phi(\overline{B}_X))$ .

## 1. INTRODUCTION AND PRELIMINARIES

The theory of measures of noncompactness has many applications in Topology, Functional analysis and Operator theory. There are many nonequivalent definitions of this notion on metric and topological spaces [1],[4]. First of them was introduced by Kuratowski in 1930. The most important examples of such functions are: Kuratowski's measure ( $\alpha$ ), Hausdorff's measure ( $\chi$ ) and measure of Istratescu ( $\beta$ ).

One of the tools that provides classification of Banach spaces considering their geometrical properties is the modulus of convexity [5]. Its natural generalization is the notion of noncompact convexity, introduced by K. Goebel and T. Sekowski [9]. Their modulus was generated with Kuratowski's measure of noncompactness ( $\alpha$ ). Modulus associated with Hausdorff's measure of noncompactness ( $\chi$ ), was introduced by Banas [3] and modulus associated with Istratescu's measure of noncompactness ( $\beta$ ) by Dominguez-Benavides and Lopez [8]. In [11], [2] was presented an abstract unified to this notions, which consider modulus of noncompact convexity  $\Delta_{X,\phi}$  associated with arbitrary abstract measure of noncompactness  $\phi$ .

Banas [3] proved that modulus  $\Delta_{X,\chi}(\varepsilon)$  is subhomogeneous function in the case of reflexive space  $X$ . Moreover, Prus [11] gave the result connecting continuity of the modulus  $\Delta_{X,\phi}(\varepsilon)$  and uniform Opial condition which imply a normal structure of the space.

In this paper, we shall prove that modulus  $\Delta_{X,\phi}(\varepsilon)$  is subhomogeneous and a continuous function on the interval  $[0, \phi(B_X))$ , for an arbitrary strictly minimalizable measure of noncompactness  $\phi$ , where  $X$  is Banach space having the Radon-Nikodym property.

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## 2. PRELIMINARIES

Let  $X$  be Banach space,  $B(x, r)$  denotes the open ball centered at  $x$  with radius  $r$ , and  $B_X$  and  $S_X$  denote the unit ball and sphere in the given space, respectively. If  $A \subset X$  with  $\overline{A}$  and  $coA$  we denote closure of the set  $A$ , that is convex hull of  $A$ .

Let  $\mathfrak{B}$  be the collection of bounded subsets of the space  $X$ . Then function  $\phi : \mathfrak{B} \rightarrow [0, +\infty)$  with properties:

- (1)  $\phi(B) = 0 \Leftrightarrow B$  is precompact set,
- (2)  $\phi(B) = \phi(\overline{B})$ ,  $\forall B \in \mathfrak{B}$ ,
- (3)  $\phi(B_1 \cup B_2) = \max\{\phi(B_1), \phi(B_2)\}$ ,  $\forall B_1, B_2 \in \mathfrak{B}$ ,

is the measure of noncompactness defined on  $X$ . For more about properties of the measure of noncompactness see in [1] and [2].

Let  $\phi$  be a measure of noncompactness. Infinite set  $A \in \mathfrak{B}$  is  $\phi$ -minimal if and only if  $\phi(A) = \phi(B)$  for any infinite set  $B \subseteq A$ .

We say that the measure of noncompactness  $\phi$  is minimalizable if for every infinite, bounded set  $A$  and for all  $\varepsilon > 0$ , there exists  $\phi$ -minimal set  $B \subset A$ , such that  $\phi(B) \geq \phi(A) - \varepsilon$ . Measure  $\phi$  is strictly minimalizable if for every infinite, bounded set  $A$ , there exists  $\phi$ -minimal set  $B \subset A$  such that  $\phi(B) = \phi(A)$ .

The modulus of noncompact convexity associated to arbitrary measure of noncompactness  $\phi$  is the function  $\Delta_{X,\phi} : [0, \phi(B_X)] \rightarrow [0, 1]$ , defined with

$$\Delta_{X,\phi}(\varepsilon) = \inf\{1 - d(0, A) : A \subseteq \overline{B_X}, A = coA = \overline{A}, \phi(A) \geq \varepsilon\}.$$

Banas [3] considered modulus  $\Delta_{X,\phi}(\varepsilon)$  for  $\phi = \chi$ , where  $\chi$  is Hausdorff measure of noncompactness. For  $\phi = \alpha$ ,  $\alpha$  is Kuratowski measure of noncompactness,  $\Delta_{X,\alpha}(\varepsilon)$  presents Gobel-Sekowski modulus of noncompact convexity [9], while for  $\phi = \beta$ , where  $\beta$  is a separation measure of noncompactness,  $\Delta_{X,\beta}(\varepsilon)$  is Dominguez Benavides -Lopez modulus of noncompact convexity [8].

The characteristic of noncompact convexity of  $X$  associated to the measure of noncompactness  $\phi$  is defined to be

$$\varepsilon_\phi(X) = \sup\{\varepsilon \geq 0 : \Delta_{X,\phi}(\varepsilon) = 0\}.$$

Inequalities

$$\Delta_{X,\alpha}(\varepsilon) \leq \Delta_{X,\beta}(\varepsilon) \leq \Delta_{X,\chi}(\varepsilon),$$

hold for moduli  $\Delta_{X,\phi}(\varepsilon)$  concerning  $\phi = \alpha, \chi, \beta$ , and consequently

$$\varepsilon_\alpha(X) \geq \varepsilon_\beta(X) \geq \varepsilon_\chi(X).$$

Banach space  $X$  has Radon-Nikodym property if and only if every nonempty, bounded set  $A \subset X$  is dentable, that is if and only if for all  $\varepsilon > 0$  there exists  $x \in A$ , such that  $x \notin \overline{co}(A \setminus \overline{B}(x, \varepsilon))$ , [6].

## 3. MAIN RESULTS

We started with our main result which is partial generalizations of earlier result obtained by Banas [3].

**Theorem 3.1.** *Let  $X$  be a Banach space with Radon-Nikodym property and  $\phi$  strictly minimalizable measure of noncompactness. The modulus  $\Delta_{X,\phi}(\varepsilon)$  is sub-homogeneous function, that is*

$$\Delta_{X,\phi}(k\varepsilon) \leq k\Delta_{X,\phi}(\varepsilon), \quad (3.1)$$

for all  $k \in [0, 1]$ ,  $\varepsilon \in [0, \phi(\overline{B}_X)]$ .

*Proof.* Let  $\eta > 0$  be arbitrary and  $\varepsilon \in [0, \phi(\overline{B}_X)]$ . From the definition of the modulus  $\Delta_{X,\phi}(\varepsilon)$  there exists convex, closed set  $A \subseteq \overline{B}_X$ ,  $\phi(A) \geq \varepsilon$  such that

$$1 - d(0, A) < \Delta_{X,\phi}(\varepsilon) + \eta. \quad (3.2)$$

The set  $kA \subseteq \overline{B}_X$  is convex and closed for arbitrary  $k \in [0, 1]$  and  $\phi(kA) = k\phi(A) \geq k\varepsilon$ . Since  $\phi$  is strictly minimalizable measure of noncompactness there is infinite  $\phi$ -minimal set  $B \subset kA$  such that

$$\phi(B) = \phi(kA) \geq k\varepsilon.$$

Let  $n_0 \in \mathbb{N}$  be such that  $\frac{k\varepsilon}{n_0} < \frac{\text{diam} B}{2}$ . Since the set  $B$  is bounded subset of Banach space  $X$  which has Radon-Nikodym property, we conclude that for  $r = \frac{k\varepsilon}{n_0}$  there exists  $x_0 \in B$  such that

$$x_0 \notin \overline{\text{co}}[B \setminus \overline{B}(x_0, r)],$$

that is

$$\overline{\text{co}}[B \setminus \overline{B}(x_0, r)] \subset B \subset kA.$$

We consider the set  $C = \overline{\text{co}}[B \setminus \overline{B}(x_0, r)]$ .  $C$  is closed and convex set and because of the properties of the strictly minimalizable measure of noncompactness we have

$$\phi(C) = \phi[B \setminus \overline{B}(x_0, r)] = \phi(B) \geq k\varepsilon.$$

Moreover,

$$1 - d(0, C) \leq 1 - d(0, kA) = 1 - kd(0, A). \quad (3.3)$$

We define the set  $B^* = C + \frac{1-k}{\|x_0\|}x_0$ .  $B^*$  is a convex and closed set and arbitrary  $z \in B^*$  is of the form  $z = y + \frac{1-k}{\|x_0\|}x_0$ , where  $y \in C \subset kA$  and  $\|z\| < 1$ . So,  $B^* \subset \overline{B}_X$ . From the properties of the measure of noncompactness  $\phi$  we have that

$$\phi(B^*) = \phi\left(C + \frac{1-k}{\|x\|}x\right) = \phi(C) \geq k\varepsilon.$$

Since  $d(0, B^*) = d(0, C) + 1 - k$ , than  $1 - d(0, B^*) = k - d(0, C) \leq k - kd(0, A)$  holds, so using inequality (3.2) we have

$$1 - d(0, B^*) < k(\Delta_{X,\phi}(\varepsilon) + \eta).$$

If we take infimum by all sets  $B$ , such that  $\phi(B) \geq k\varepsilon$ , we have

$$\Delta_{X,\phi}(k\varepsilon) \leq k\Delta_{X,\phi}(\varepsilon) + k\eta.$$

Since  $\eta > 0$  can be choosen arbitrarily small we obtain

$$\Delta_{X,\phi}(k\varepsilon) \leq k\Delta_{X,\phi}(\varepsilon).$$

□

As applications of Theorem we shall state the following corollaries.

**Corollary 3.2.** *Let  $\phi$  be a strictly minimalizable measure of noncompactness defined on space  $X$  with Radon-Nikodym property. The function  $\Delta_{X,\phi}(\varepsilon)$  is strictly increasing on the interval  $[\varepsilon_\phi(X), \phi(\overline{B}_X)]$ .*

*Proof.* Let  $\varepsilon_1, \varepsilon_2 \in [\varepsilon_\phi(X), \phi(\overline{B}_X)]$  and  $\varepsilon_1 < \varepsilon_2$ . If we put  $k = \frac{\varepsilon_1}{\varepsilon_2} < 1$ , then by the Theorem 3 we obtain

$$\Delta_{X,\phi}(\varepsilon_1) = \Delta_{X,\phi}(k\varepsilon_2) \leq k\Delta_{X,\phi}(\varepsilon_2) < \Delta_{X,\phi}(\varepsilon_2) .$$

□

**Corollary 3.3.** *Let  $\phi$  be a strictly minimalizable measure of noncompactness defined on space  $X$  with Radon-Nikodym property. Inequality*

$$\Delta_{X,\phi}(\varepsilon) \leq \varepsilon$$

*holds for all  $\varepsilon \in [0, \phi(\overline{B}_X)]$ .*

*Proof.* If  $\varepsilon \in [0, 1]$  and if  $k$  and  $\varepsilon$  change roles, and  $\varepsilon$  take the value  $\varepsilon = 1$ , than by using Theorem 3 we have

$$\Delta_{X,\phi}(\varepsilon) \leq \varepsilon\Delta_{X,\phi}(1) \leq \varepsilon .$$

If  $\varepsilon \in (1, \phi(\overline{B}_X)]$ , than the monotonicity of the modulus  $\Delta_{X,\phi}(\varepsilon)$  provides that

$$\Delta_{X,\phi}(\varepsilon) \leq \Delta_{X,\phi}(\phi(\overline{B}_X)) = 1 < \varepsilon .$$

□

**Corollary 3.4.** *Let  $\phi$  be a strictly minimalizable measure of noncompactness defined on the space  $X$  with Radon-Nikodym property. The function  $f(\varepsilon) = \frac{\Delta_{X,\phi}(\varepsilon)}{\varepsilon}$  is nondecreasing on the interval  $[0, \phi(\overline{B}_X)]$  and for  $\varepsilon_1 + \varepsilon_2 \leq \phi(\overline{B}_X)$  it holds*

$$\Delta_{X,\phi}(\varepsilon_1 + \varepsilon_2) \geq \Delta_{X,\phi}(\varepsilon_1) + \Delta_{X,\phi}(\varepsilon_2). \quad (3.4)$$

*Proof.* Let  $\varepsilon_1, \varepsilon_2 \in [0, \phi(\overline{B}_X)]$  such that  $\varepsilon_1 \leq \varepsilon_2$ . We put  $k = \frac{\varepsilon_1}{\varepsilon_2}$ . Then

$$f(\varepsilon_1) = \frac{\Delta_{X,\phi}(\varepsilon_1)}{\varepsilon_1} = \frac{\Delta_{X,\phi}(k\varepsilon_2)}{\varepsilon_1} .$$

If we use a property of subhomegeneity of the function  $\Delta_{X,\phi}(\varepsilon)$  we have

$$f(\varepsilon_1) \leq \frac{\Delta_{X,\phi}(\varepsilon_2)}{\varepsilon_2} = f(\varepsilon_2) ,$$

which proves that  $f(\varepsilon)$  is a nondecreasing function on the interval  $[0, \phi(B_X)]$ . Furthermore

$$\begin{aligned}
\Delta_{X,\phi}(\varepsilon_1) + \Delta_{X,\phi}(\varepsilon_2) &\leq k\Delta_{X,\phi}(\varepsilon_2) + \Delta_{X,\phi}(\varepsilon_2) \\
&= \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_2} \Delta_{X,\phi}(\varepsilon_2) \\
&\leq (\varepsilon_1 + \varepsilon_2) \frac{\Delta_{X,\phi}(\varepsilon_1 + \varepsilon_2)}{\varepsilon_1 + \varepsilon_2} \\
&= \Delta_{X,\phi}(\varepsilon_1 + \varepsilon_2).
\end{aligned}$$

So, inequality (3.4) is proved.  $\square$

**Corollary 3.5.** *Let  $\phi$  be a strictly minimalizable measure of noncompactness defined on the space  $X$  with Radon-Nikodym property.*

$$\frac{\Delta_{X,\phi}(\varepsilon_2) - \Delta_{X,\phi}(\varepsilon_1)}{\varepsilon_2 - \varepsilon_1} \geq \frac{\Delta_{X,\phi}(\varepsilon_2)}{\varepsilon_2}. \quad (3.5)$$

holds for all  $\varepsilon_1, \varepsilon_2 \in (\varepsilon_1(X), \phi(\overline{B}_X)]$ ,  $\varepsilon_1 \leq \varepsilon_2$ .

*Proof.* Let  $k = \frac{\varepsilon_1}{\varepsilon_2} \leq 1$ . From the Theorem 3 we have

$$\begin{aligned}
\Delta_{X,\phi}(\varepsilon_2) - \Delta_{X,\phi}(\varepsilon_1) &= \Delta_{X,\phi}(\varepsilon_2) - \Delta_{X,\phi}(k\varepsilon_2) \\
&\geq \Delta_{X,\phi}(\varepsilon_2) - k\Delta_{X,\phi}(\varepsilon_2) \\
&= \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2} \Delta_{X,\phi}(\varepsilon_2).
\end{aligned}$$

Thus the inequality (3.5) is proved.  $\square$

#### 4. CONTINUITY OF THE MODULUS OF NONCOMPACT CONVEXITY

In this section we shall prove continuity of the modulus of noncompact convexity associated with arbitrary strictly minimalizable measure of noncompactness, defined on Banach space with Radon-Nikodym property. Now we need the following Lemmas.

**Lemma 4.1.** *Let  $\phi$  be an arbitrary measure of noncompactness,  $B \subset \overline{B}_X$   $\phi$ -minimal set and  $k > 0$  arbitrary. Then the set  $kB$  is  $\phi$ -minimal.*

*Proof.* Let  $B$  be  $\phi$ -minimal subset of closed unit ball and  $D \subset kB$  arbitrary infinite set for  $k > 0$ . If  $y \in \frac{1}{k}D$ , then  $y = \frac{1}{k}y'$  for some  $y' \in D$ . Since  $y' \in kB$ , then  $y' = kx$  for some  $x \in B$ . So  $y = \frac{1}{k}kx = x \in B$ , therefore  $\frac{1}{k}D \subset B$ . Since  $B$  is  $\phi$ -minimal set we have

$$\phi(B) = \phi\left(\frac{1}{k}D\right) = \frac{1}{k}\phi(D),$$

that is  $\phi(D) = k\phi(B) = \phi(kB)$ . Thus  $kB$  is  $\phi$ -minimal set.  $\square$

**Lemma 4.2.** *Let  $X$  be a Banach space with Radon-Nikodym property and  $\phi$  a strictly minimalizable measure of noncompactness. The modulus of noncompact convexity  $\Delta_{X,\phi}(\varepsilon)$  is left continuous function on the interval  $[0, \phi(\overline{B}_X))$ .*

*Proof.* Let  $\varepsilon_0 \in [0, \phi(\overline{B}_X))$  be arbitrary and let  $\varepsilon < \varepsilon_0$ . From the definition of  $\Delta_{X,\phi}(\varepsilon)$ , for arbitrary  $\eta > 0$  there exists convex and closed set  $A \subset \overline{B}_X$ ,  $\phi(A) \geq \varepsilon$  such that

$$1 - d(0, A) < \Delta_{X,\phi}(\varepsilon) + \eta .$$

Since  $\phi$  is strictly minimalizable measure of noncompactness there exists infinite  $\phi$ -minimal set  $B \subset A$  such that  $\phi(A) = \phi(B) \geq \varepsilon$ . Moreover, inequality

$$1 - d(0, B) \leq 1 - d(0, A) < \Delta_{X,\phi}(\varepsilon) + \eta$$

holds for the set  $B$ . Let  $n_0 \in \mathbb{N}$  be such that  $\frac{\varepsilon}{n_0} < \frac{\text{diam} B}{2}$ . Since  $B$  is a bounded subset of the Banach space  $X$  with Radon-Nikodym property ([6]) we conclude that for  $r = \frac{\varepsilon}{n_0}$  there exists  $x_0 \in B$  such that

$$x_0 \notin \overline{co} [B \setminus \overline{B}(x_0, r)] .$$

This means that there is a convex and closed set  $C = \overline{co} [B \setminus \overline{B}(x_0, r)] \subset A$ , where

$$1 - d(0, C) \leq 1 - d(0, A) < \Delta_{X,\phi}(\varepsilon) + \eta .$$

Since  $\phi(B) \geq \varepsilon$  we have that  $B \setminus \overline{B}(x_0, r)$  is an infinite set and using properties of the strictly minimalizable measure of noncompactness  $\phi$  we obtain

$$\phi(C) = \phi [B \setminus \overline{B}(x_0, r)] = \phi(B) \geq \varepsilon .$$

Let  $k = 1 + \frac{1 - d(0, C)}{2}$ . We will consider set  $A^* = kC \cap \overline{B}_X$ .  $A^*$  is a closed and convex set such that  $A^* \subseteq kC \subset kB$ . From the Lemma 4.1 we have

$$\phi(A^*) = \phi(kB) = k\phi(B) \geq k\varepsilon .$$

Moreover, inequality

$$1 - d(0, A^*) \leq 1 - d(0, kC) < 1 - d(0, C) ,$$

holds, i.e.

$$1 - d(0, A^*) < \Delta_{X,\phi}(\varepsilon) + \eta . \tag{4.1}$$

Let  $\delta = \varepsilon_0 \left(1 - \frac{1}{k}\right)$ . For  $\varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0)$  we have

$$\phi(A^*) \geq k\varepsilon > k(\varepsilon_0 - \delta) = k\frac{\varepsilon_0}{k} = \varepsilon_0 .$$

If we take infimum in (4.1) by all the sets  $A^*$  such that  $\phi(A^*) > \varepsilon_0$ , we obtain

$$\inf\{1 - d(0, A^*) : A^* \subset \overline{B}_X, A^* = \overline{co}(A^*), \phi(A^*) > \varepsilon_0\} \leq \Delta_{X,\phi}(\varepsilon) + \eta ,$$

that is

$$\Delta_{X,\phi}(\varepsilon_0) \leq \Delta_{X,\phi}(\varepsilon) + \eta .$$

Since  $\eta > 0$  was arbitrarily small we conclude  $\lim_{\varepsilon \rightarrow \varepsilon_0^-} \Delta_{X,\phi}(\varepsilon) = \Delta_{X,\phi}(\varepsilon_0)$ .  $\square$

**Lemma 4.3.** *Let  $X$  be a Banach space with Radon-Nikodym property and  $\phi$  a strictly minimalizable measure of noncompactness. The function  $\Delta_{X,\phi}(\varepsilon)$  is right continuous on the interval  $[0, \phi(\overline{B}_X))$ .*

*Proof.* Let  $\eta > 0$  and  $\varepsilon_0 \in [0, \phi(\overline{B}_X))$ . From the definition of the modulus of noncompact convexity  $\Delta_{X,\phi}(\varepsilon)$ , there exists convex and closed set  $A \subset \overline{B}_X$ ,  $\phi(A) \geq \varepsilon_0$ , such that

$$1 - d(0, A) < \Delta_{X,\phi}(\varepsilon_0) + \eta .$$

Since  $\phi$  is strictly minimalizable measure we conclude that there exists infinite,  $\phi$ -minimal set  $B \subset A$  such that  $\phi(A) = \phi(B) \geq \varepsilon_0$  and

$$1 - d(0, B) \leq 1 - d(0, A) < \Delta_{X,\phi}(\varepsilon_0) + \eta .$$

The set  $B$  is bounded subset of the Banach space that has Radon-Nikodym property, and because of that for  $r = \frac{\varepsilon_0}{n_0}$ , where  $n_0 \in \mathbb{N}$  is such that  $\frac{\varepsilon_0}{n_0} < \frac{\text{diam} B}{2}$  we can find  $x_0 \in B$  such that

$$x_0 \notin \overline{co} [B \setminus \overline{B}(x_0, r)] .$$

This implies that there is convex and closed set  $C = \overline{co} [B \setminus \overline{B}(x_0, r)] \subset A$ , where

$$1 - d(0, C) \leq 1 - d(0, A) < \Delta_{X,\phi}(\varepsilon_0) + \eta .$$

Hence, by the properties of the strictly minimalizable measure of noncompactness  $\phi$  we obtain

$$\phi(C) = \phi [B \setminus \overline{B}(x_0, r)] = \phi(B) \geq \varepsilon_0 .$$

- (1) If  $\phi(C) > \varepsilon_0$ , let  $\delta = \phi(C) - \varepsilon_0$  and consider arbitrary  $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \delta)$ . Then  $\phi(C) > \varepsilon$ , hence

$$\inf \{1 - d(0, C) : C \subset \overline{B}_X, C = \overline{co} C, \phi(C) > \varepsilon\} \leq \Delta_{X,\phi}(\varepsilon_0) + \eta .$$

Moreover,

$$\Delta_{X,\phi}(\varepsilon) \leq \Delta_{X,\phi}(\varepsilon_0) + \eta .$$

This completes the proof of the Theorem.

- (2) Let  $\phi(C) = \varepsilon_0$ . Consider the set  $B^* = kC \cap \overline{B}_X$  for  $k = 1 + \frac{1 - d(0, C)}{2}$ .  $B^*$  is closed, convex set and it holds that  $B^* \subseteq kC \subset kB$ . Hence by the Lemma 4.1 we have

$$\phi(B^*) = \phi(kB) = k\phi(B) = k\varepsilon_0 .$$

Moreover, next inequality holds

$$1 - d(0, B^*) \leq 1 - d(0, kC) < 1 - d(0, C) .$$

Let  $\delta = \varepsilon_0(1 - k)$ . Then for  $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \delta)$  we have  $\phi(B^*) = k\varepsilon_0 > \varepsilon$ , thus  $\inf \{1 - d(0, B^*) : B^* \subset \overline{B}_X, B^* = \overline{co} B^*, \phi(B^*) > \varepsilon\} \leq \Delta_{X,\phi}(\varepsilon_0) + \eta ,$

that is

$$\Delta_{X,\phi}(\varepsilon) \leq \Delta_{X,\phi}(\varepsilon_0) + \eta .$$

Therefore, we obtain  $\lim_{\varepsilon \rightarrow \varepsilon_0+} \Delta_{X,\phi}(\varepsilon) = \Delta_{X,\phi}(\varepsilon_0)$ . This completes the proof.  $\square$

From Lemma 4.2 and Lemma 4.3 follows that:

**Theorem 4.4.** *Let  $X$  be a Banach space with Radon-Nikodym property and  $\phi$  a strictly minimalizable measure of noncompactness. The modulus  $\Delta_{X,\phi}(\varepsilon)$  is a continuous function on the interval  $[0, \phi(\overline{B}_X))$ .*

It is known that the Hausdorff measure of noncompactness  $\chi$  is a strictly minimalizable measure of noncompactness in the weakly compactly generated Banach spaces ([2], Theorem III. 2.7.). Since reflexive spaces are weakly compactly generated and also have Radon-Nikodym property [10], we conclude that the modulus of noncompact convexity  $\Delta_{X,\chi}$ , associated to measure of noncompactness  $\chi$ , is a continuous function on the interval  $[0, \phi(\overline{B}_X))$  in an arbitrary weakly compactly generated space  $X$ .

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